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Random matrix ensembles with an effective extensive external charge

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Abstract. Recent theoretical studies of chaotic scattering have encountered ensembles of random matrices in which the eigenvalue probability density function contains a one-body factor with an exponent proportional to the number of eigenvalues. Two such ensembles have been encountered; an ensemble of unitary matrices specified by the so-called Poisson kernel, and the Laguerre ensemble of positive definite matrices. Here, we consider various properties of these ensembles. Jack polynomial theory is used to prove a reproducing property of the Poisson kernel, and a certain unimodular mapping is used to demonstrate that the variance of a linear statistic is the same as in the Dyson circular ensemble. For the Laguerre ensemble, the scaled global density is calculated exactly for all even values of the parameter β, while for $\beta = 2$ (random matrices with unitary symmetry), the neighbourhood of the smallest eigenvalue is shown to be in the soft edge universality class.

1. Introduction

In the theory of random matrices, a primary task is to compute the probability density function (PDF) for the eigenvalues from knowledge of the PDF for the ensemble of matrices. Two examples of random matrix ensembles of interest in this paper are Dyson's [Dys62] circular ensembles of symmetric, unitary and self-dual quaternion unitary random matrices (labelled by $\beta = 1$, 2 and 4 respectively), and the Laguerre ensemble of random Wishart matrices $A = X^\dagger X$, where X is a random $(M \times N)$ matrix $(M \geqslant N)$ which has either real ($\beta = 1$), complex ($\beta = 2$) or quaternion real ($\beta = 4$) Gaussian random elements. In the circular ensemble, the PDF for the matrices is uniquely specified by requiring that it be uniform and unchanged by mappings of the form $U \mapsto VUV'$, where V is an arbitrary $(N \times N)$ unitary matrix and $V' = V^{\mathrm{T}}$ for $\beta = 1$, V' is arbitrary for $\beta = 2$ and $V' = V^{\mathrm{D}}$ for $\beta = 4$ (D denotes the quaternion dual). In the Laguerre ensemble, the distribution of the elements of X are taken to be the Gaussian $A\,\mathrm{e}^{-\beta \mathrm{Tr}(X^\dagger X)/2}$ which is equivalent to choosing each element independently with a Gaussian distribution $A'\,\mathrm{e}^{-\beta|x_{jk}|^2/2}$.

The corresponding PDF for the eigenvalues $\mathrm{e}^{\mathrm{i}\theta_j}$, $j = 1, \ldots, N$ in the circular ensemble is

$$\frac{1}{C_{\beta N}} \prod_{1\leqslant j<k\leqslant N} |\mathrm{e}^{\mathrm{i}\theta_k} - \mathrm{e}^{\mathrm{i}\theta_j}|^\beta \tag{1.1}$$

while for the Laguerre ensemble the PDF is given by

$$\frac{1}{C'_{a\beta N}} \prod_{j=1}^{N} \mathrm{e}^{-\beta\lambda_j/2} \lambda_j^{a\beta/2} \prod_{1\leqslant j<k\leqslant N} |\lambda_k - \lambda_j|^\beta \qquad \lambda_j > 0 \tag{1.2}$$

where $a := M - N + 1 - 2/\beta$. In both cases the eigenvalue PDFs can be written in the form of a Boltzmann factor for a classical gas, with potential energy consisting of one- and two-body terms only and interacting at inverse temperature β

$$\exp\left(-\beta\left(\sum_{j=1}^{N} V_1(x_j) + \sum_{1\leqslant j<k\leqslant N} V_2(x_j, x_k)\right)\right). \tag{1.3}$$

Thus, for the circular ensemble

$$V_1(x) = 0 \qquad V_2(x, y) = -\log|\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{\mathrm{i}y}| \tag{1.4}$$

while for the Laguerre ensemble

$$V_1(x) = \frac{x}{2} - \frac{a}{2}\log x \qquad V_2(x, y) = -\log|x - y| \tag{1.5}$$

which displays the well known fact that the analogous classical gas has a two-body logarithmic potential. The logarithmic potential is special in that it is the Coulomb potential between like charges in two dimensions. In (1.4) the two-dimensional charges are confined to a unit circle, while in (1.5) the two-dimensional charges are free to move on the half line $x > 0$, but are confined to the neighbourhood of the origin by the one-body potential.

In this work we will focus attention on a sub-class of random matrix ensembles with eigenvalue PDFs of the form (1.3) in which the one-body potential contains a term proportional to $N\log|1 - \mu^*\,\mathrm{e}^{\mathrm{i}x}|$ (unitary matrices) or $-N\log|x|$ (Laguerre ensemble). Again the two body term is logarithmic. In the classical gas these one-body potentials can be interpreted as being due to an external fixed charge at the point $1/\mu^*$ in the complex plane with strength proportional to $-N$ (unitary matrices), and as an external charge fixed at the origin of strength proportional to N (Laguerre ensemble). We see from (1.2) that an example of a random matrix of this type in the Laguerre case is a Wishart matrix with X a rectangular matrix in which the number of columns is some fixed fraction of the number of rows. In the theory of random unitary matrices, this type of eigenvalue PDF results as a special case of the the ensemble of random matrices defined by the Poisson kernel

$$\frac{1}{C}\frac{\det(1 - \bar{S}\bar{S}^\dagger)^{\beta(N-1)/2+1}}{|\det(1 - \bar{S}^\dagger S)|^{\beta(N-1)+2}} \tag{1.6}$$

where the notation $\bar{S}$ denotes the average of S. Random unitary matrices with this PDF occur in the study of scattering problems in nuclear physics [MPS85] and mesoscopic systems [Bro95, FS97]. In the case $\bar{S} = 0$, this reduces to the PDF specifying Dyson's circular ensemble (all members equally probable). In the case that $\bar{S} = \mu 1_N$, $|\mu| < 1$ (1_N denotes the $(N \times N)$ unit matrix), the corresponding eigenvalue PDF is given by

$$\frac{1}{C_{\beta N}}\prod_{j=1}^{N}\frac{(1 - |\mu|^2)^{\beta a'/2}}{|1 - \mu^*\,\mathrm{e}^{\mathrm{i}\theta}|^{\beta a'}}\prod_{1\leqslant j<k\leqslant N}|\mathrm{e}^{\mathrm{i}\theta_k} - \mathrm{e}^{\mathrm{i}\theta_j}|^{\beta} \tag{1.7}$$

where $a' = (N - 1 + 2/\beta)$ and $C_{\beta N}$ is as in (1.1), and so in the classical gas picture there is a fixed charge of opposite sign at the point $1/\mu^*$, and the magnitude of this charge is indeed proportional to N.

In section 2 we will consider the eigenvalue PDF (1.7) for the Poisson kernel. First, the special reproducing property of the PDFs (1.6) and (1.7) for $\beta = 1, 2$ and 4 will be revised, and extended to all $\beta > 0$ in the case of (1.7). Our tool here is Jack polynomial theory [Mac95]. Then, we will consider the effect of the N-dependent exponent in (1.7) on the one- and two-body correlation functions, as well as on the fluctuation formula for the variance of a linear statistic.

In section 3 a physical problem giving rise to the Laguerre ensemble in which a in (1.2) is equal to N will be revised. Then we will revise known theorems in the cases $\beta = 1$ and 2 for the global density and the distribution of the largest and smallest eigenvalues. Next, known integral formulae for the density [BF97a] at general even β, deduced from the theory of generalized Selberg integrals [Kan93] and their relationship to Jack polynomial theory, will be analysed in the appropriate limit to deduce that the formula for the global density holds independently of β. For the special coupling $\beta = 2$ the local distribution functions in the neighbourhood of the smallest eigenvalue are analysed and shown to belong to the universality class of the soft edge, giving rise to the Airy kernel [For93a, TW94a, KF97b]. We conclude the section with an analysis of some nonlinear equations [TW94a, TW94b], which explicitly demonstrate the universality of the distribution of the smallest eigenvalue.

2. The Poisson kernel

2.1. Physical origin of the Poisson kernel

The scattering matrix for N channels entering and leaving a chaotic cavity via a non-ideal lead containing a tunnel barrier has as its PDF the Poisson kernel (1.6) [Bro95]. Also, in scattering problems in nuclear physics, (1.6) has been used [MPS85] to describe situations in which the average of the scattering matrix is non-zero. It was in the latter problem that the Poisson kernel first appeared in an application of random matrix theory. In [MPS85] a requirement for the PDF of the ensemble of scattering matrices was the special reproducing property

$$f(\bar{S}) := f(\langle S\rangle) = \langle f(S)\rangle \tag{2.1}$$

for f analytic in S. It was noted that a result of Hua [Hua63] gives that

$$f(\bar{S}) = \frac{1}{C}\int f(S)\frac{\det(1_N - \bar{S}\bar{S}^\dagger)^{\beta(N-1)/2+1}}{|\det(1_N - \bar{S}^\dagger S)|^{\beta(N-1)+2}}\mu(\mathrm{d}S) \tag{2.2}$$

where $\mu(\mathrm{d}S)$ is the invariant measure associated with the Dyson circular ensemble, and thus that the Poisson kernel exhibits the reproducing property (2.1).

2.2. The reproducing property for general β

To make any sense out of (2.2) it is necessary that $\beta = 1, 2$ or 4, so that the measure $\mu(\mathrm{d}S)$ has meaning. However, in the case $\bar{S} = \mu 1_N$, the corresponding eigenvalue PDF (1.7) can be interpreted as a Boltzmann factor and it makes sense to consider all $\beta > 0$. In this case, for $\beta = 1, 2$ and 4, (1.7) used in (2.2) gives that the reproducing property restricted to analytic functions of the eigenvalues $\mathrm{e}^{\mathrm{i}\theta_j}$ reads

$$f(\mu,\dots,\mu) = \frac{1}{C_{\beta N}}\prod_{l=1}^{N}\int_0^{2\pi}\mathrm{d}\theta_l\frac{(1-|\mu|^2)^{\beta a'/2}}{|1-\mu^*\,\mathrm{e}^{\mathrm{i}\theta_l}|^{\beta a'}}f(\mathrm{e}^{\mathrm{i}\theta_1},\dots,\mathrm{e}^{\mathrm{i}\theta_N})\prod_{1\leqslant j<k\leqslant N}|\mathrm{e}^{\mathrm{i}\theta_k}-\mathrm{e}^{\mathrm{i}\theta_j}|^\beta. \tag{2.3}$$

This equation is well defined for all $\beta > 0$, but its validity has only been established for $\beta = 1, 2$ and 4. Here we will establish its validity for general $\beta > 0$, using the properties of the orthogonal polynomials associated with the PDF (1.1). Note that in the case $N = 1$ (2.3) reads

$$f(\mu) = \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-|\mu|^2)}{|1-\mu^*\,\mathrm{e}^{\mathrm{i}\theta}|^2}f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \tag{2.4}$$

which is the celebrated Poisson formula on a circle, giving the value of an analytic function f for $|\mu| < 1$ in terms of its value on the unit circle. Thus (2.3) can be regarded as an N-dimensional generalization of this result.

The multivariable orthogonal polynomials corresponding to (1.1) are the symmetric Jack polynomials $P_\kappa^{(2/\beta)}(z)$ [Sta89, Mac95], as they possess the orthogonality property

$$\prod_{l=1}^{N}\int_0^{2\pi} \mathrm{d}\theta_l\, P_\kappa^{(2/\beta)}(z) P_\sigma^{(2/\beta)}(z^*) \prod_{1\leqslant j<k\leqslant N} |\mathrm{e}^{\mathrm{i}\theta_k} - \mathrm{e}^{\mathrm{i}\theta_j}|^\beta = \mathcal{N}_\kappa \delta_{\kappa,\sigma} \tag{2.5}$$

and form a complete set for the space of analytic functions. Here the labels κ and σ are partitions consisting of N parts or less, and $z := (\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_N})$. The normalization $\mathcal{N}_\kappa$ is given by

$$\frac{\mathcal{N}_\kappa}{\mathcal{N}_0} = \frac{P_\kappa^{(2/\beta)}(1^N) d'_\kappa}{[1+\beta(N-1)/2]_\kappa^{(2/\beta)}} \tag{2.6}$$

where

$$d'_\kappa := \prod_{s\in\kappa}\left(\frac{2}{\beta}a(s) + l(s) + 1\right) \qquad [u]_\kappa^{(\alpha)} := \prod_{j=1}^{N} \frac{\Gamma(u - (\beta/2)(j-1) + \kappa_j)}{\Gamma(u - (\beta/2)(j-1))}.$$

The notation $s \in \kappa$ refers to the diagram of the partition κ, and $a(s) = \kappa_i - j$ is the arm length while $l(s) = \kappa'_j - i$ is the leg length (κ' refers to the conjugate partition of κ); see for example [Mac95]. The normalization $\mathcal{N}_0$ is the same quantity as the normalization $C_{\beta N}$ in (1.1), and has the explicit value

$$\mathcal{N}_0 = C_{\beta N} = (2\pi)^N \frac{(N\beta/2)!}{(\beta/2)!^N}. \tag{2.7}$$

The quantity $P_\kappa^{(2/\beta)}(1^N)$ in (2.6) also has an explicit evaluation [Sta89, Mac95, BF97b], but it suits our purposes to leave it unevaluated.

With these preliminaries, we now pose the problem of specifying the kernel $K(w, z^*)$, $w = (\mathrm{e}^{\mathrm{i}\phi_1}, \ldots, \mathrm{e}^{\mathrm{i}\phi_N})$, $\mathrm{Im}(\phi_j) > 0$, which is an analytic function of z^*, and has the reproducing property

$$f(w) = \prod_{l=1}^{N}\int_0^{2\pi} \mathrm{d}\theta_l\, K(w, z^*) f(z) \prod_{1\leqslant j<k\leqslant N} |\mathrm{e}^{\mathrm{i}\theta_k} - \mathrm{e}^{\mathrm{i}\theta_j}|^\beta \tag{2.8}$$

for f analytic and symmetric in $w_1, \ldots, w_N$. The proof of (2.3) will then consist first of evaluating $K(w, z^*)$ at $w = (\mu, \ldots, \mu)$, then transforming equation (2.8) so that $K(w, z^*)$ becomes real. Using the orthogonality property (2.5), as well as the completeness of $\{P_\kappa^{(2/\beta)}(z)\}$ for analytic functions, it is a simple exercise to check that the required kernel K is uniquely given by

$$\begin{aligned} K(w, z^*) &= \sum_\kappa \frac{P_\kappa^{(2/\beta)}(w) P_\kappa^{(2/\beta)}(z^*)}{\mathcal{N}_\kappa} \\ &= \frac{1}{\mathcal{N}_0} \sum_\kappa [1+\beta(N-1)/2]_\kappa^{(2/\beta)} \frac{P_\kappa^{(2/\beta)}(w) P_\kappa^{(2/\beta)}(z^*)}{d'_\kappa P_\kappa^{(2/\beta)}(1^N)} \end{aligned} \tag{2.9}$$

where the second equality follows from (2.6). However, in general, the generalized hypergeometric function ${}_1\mathcal{F}_0^{(2/\beta)}(a; w, z^*)$ is defined by

$${}_1\mathcal{F}_0^{(2/\beta)}(a; w, z^*) = \sum_\kappa [a]_\kappa^{(2/\beta)} \frac{P_\kappa^{(2/\beta)}(w) P_\kappa^{(2/\beta)}(z^*)}{d'_\kappa P_\kappa^{(2/\beta)}(1^N)} \tag{2.10}$$

so we have

$$K(w, z^*) = \frac{1}{\mathcal{N}_0} {}_1\mathcal{F}_0^{(2/\beta)}(1 + \beta(N-1)/2; w; z^*). \tag{2.11}$$

In general, there is no known expression for ${}_1\mathcal{F}_0^{(2/\beta)}(a; w, z^*)$ in terms of elementary functions. However, the case $w = (\mu, \dots, \mu)$ is an exception, for then we have

$${}_1\mathcal{F}_0^{(2/\beta)}(a; w, z^*)|_{w=(\mu,\dots,\mu)} = {}_1\mathcal{F}_0^{(2/\beta)}(a; \mu z^*) := \sum_\kappa \frac{\mu^{|\kappa|}[a]_\kappa^{(2/\beta)} P_\kappa^{(2/\beta)}(z^*)}{d'_\kappa}. \tag{2.12}$$

The significance of this is that ${}_1\mathcal{F}_0^{(2/\beta)}(a; \mu z^*)$ can be summed according to the generalized binomial formula [Kan93]

$${}_1\mathcal{F}_0^{(2/\beta)}(a; \mu z^*) = \prod_{j=1}^N \frac{1}{(1-\mu z_j^*)^a} \qquad |\mu z_j| < 1. \tag{2.13}$$

Comparing (2.11)–(2.13) we therefore have

$$K(w, z^*)|_{w=(\mu,\dots,\mu)} = \frac{1}{\mathcal{N}_0} \prod_{j=1}^N \frac{1}{(1-\mu z_j^*)^{1+\beta(N-1)/2}}. \tag{2.14}$$

Although this is an explicit solution to the problem of determining the kernel in (2.8) in the case where $w = (\mu, \dots, \mu)$, it does not immediately establish (2.3) as the kernel (2.14) is not real. Note that in the case $N = 1$, (2.14) is the Cauchy kernel from elementary complex analysis. For general N, to obtain a real (Poisson) kernel from the Cauchy kernel, we proceed as in the one-dimensional case and simply make the replacement

$$f \mapsto \prod_{j=1}^N \frac{1}{(1-\mu^* z_j)^{1+\beta(N-1)/2}} f$$

in (2.8) with $w = (\mu, \dots, \mu)$. Formula (2.3) results, thus establishing its validity for general $\beta > 0$.

2.3. Fluctuation formulae

In the application of random matrix theory, an important class of observables are the linear statistics $A = \sum_{j=1}^N a(\lambda_j)$. The first two moments of these statistics are given by

$$\langle A \rangle = \int_0^{2\pi} \rho_{(1)}(\theta) a(\theta)\, \mathrm{d}\theta \tag{2.15}$$

$$\mathrm{Var}(A) := \int_0^{2\pi} \mathrm{d}\theta_1\, a(\theta_1) \int_0^{2\pi} \mathrm{d}\theta_2\, a(\theta_2) S(\theta_1, \theta_2) \tag{2.16}$$

where $S(\lambda_1, \lambda_2)$ denotes the structure function

$$S(\lambda_1, \lambda_2) := \rho_{(2)}^{\mathrm{T}}(\lambda_1, \lambda_2) + \rho_{(1)}(\lambda_1)\delta(\lambda_1 - \lambda_2)$$

with $\rho_{(1)}$ denoting the density and $\rho_{(2)}^{\mathrm{T}}$ denoting the truncated two-particle distribution function. In this subsection these quantities will be considered for the eigenvalue PDF (1.7).

It is instructive to first consider the case where $\mu = 0$, when (1.7) reduces to the eigenvalue PDF (1.1) for the circular ensemble. In this case [For95]

$$\rho_{(1)}(\theta^{(0)}) = \frac{N}{2\pi} \tag{2.17}$$

$$S(\theta_1^{(0)}, \theta_2^{(0)}) \underset{N\to\infty}{\sim} -\frac{1}{\beta\pi^2}\frac{\partial^2}{\partial\theta_1^{(0)}\partial\theta_2^{(0)}} \log|\sin(\theta_1^{(0)} - \theta_2^{(0)})/2| \tag{2.18}$$

(the use of the superscript (0) indicates that $\mu = 0$ in (1.7)), where in the asymptotic expression (2.18) all oscillatory terms, each of which have a period some integer multiple of $2\pi/N$, are ignored. For example, we have the exact result (see e.g. [Meh91])

$$\begin{aligned}\rho_{(2)}^{\mathrm{T}}(\theta_1^{(0)}, \theta_2^{(0)}) &= -\left(\frac{1}{2\pi}\right)^2 \frac{\sin^2 N(\theta_1^{(0)} - \theta_2^{(0)})/2}{\sin^2(\theta_1^{(0)} - \theta_2^{(0)})/2} \\ &= -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2\left(\frac{1}{\sin^2(\theta_1^{(0)} - \theta_2^{(0)})/2} - \frac{\cos N(\theta_1^{(0)} - \theta_2^{(0)})}{\sin^2(\theta_1^{(0)} - \theta_2^{(0)})/2}\right).\end{aligned}$$

Ignoring the oscillatory term with the factor of $\cos N(\theta_1^{(0)} - \theta_2^{(0)})$ gives

$$\rho_{(2)}^{\mathrm{T}}(\theta_1^{(0)}, \theta_2^{(0)}) \underset{N\to\infty}{\sim} -\frac{1}{2(2\pi)^2}\frac{1}{\sin^2(\theta_1^{(0)} - \theta_2^{(0)})/2} = -\frac{1}{2\pi^2}\frac{\partial^2}{\partial\theta_1^{(0)}\partial\theta_2^{(0)}} \log|\sin(\theta_1^{(0)} - \theta_2^{(0)})/2|$$

valid for $\theta_1^{(0)} \neq \theta_2^{(0)}$. The validity of (2.18) for $\theta_1^{(0)} = \theta_2^{(0)}$ is then deduced by noting that if we write (define)

$$\int_0^{2\pi} \frac{\partial^2}{\partial\theta_1^{(0)}\partial\theta_2^{(0)}} \log|\sin(\theta_1^{(0)} - \theta_2^{(0)})/2| \, \mathrm{d}\theta_1^{(0)} = \frac{\partial}{\partial\theta_2^{(0)}} \int_0^{2\pi} \frac{\partial}{\partial\theta_1^{(0)}} \log|\sin(\theta_1^{(0)} - \theta_2^{(0)})/2| \, \mathrm{d}\theta_1^{(0)} \tag{2.19}$$

then this integral vanishes. This is the perfect screening sum rule for the underlying log-gas system, and is a fundamental requirement of $S(\theta_1^{(0)}, \theta_2^{(0)})$ [Mar88].

Substituting (2.18) in (2.16), interchanging the order of differentiation and integration according to (2.19), and using the Fourier expansion

$$\log|\sin(\theta_1^{(0)} - \theta_2^{(0)})/2| = \sum_{p=-\infty}^{\infty} \alpha_p \, \mathrm{e}^{\mathrm{i}p(\theta_1^{(0)} - \theta_2^{(0)})}$$

$$\alpha_p = -\frac{1}{2|p|} \quad (p \neq 0) \qquad \alpha_0 = -2\pi \log 2$$

allows (2.16) to be evaluated as

$$\mathrm{Var}(A)^{(0)} = \frac{4}{\beta}\sum_{n=1}^{\infty} n a_n a_{-n} \qquad a(\theta) = \sum_{n=-\infty}^{\infty} a_n \, \mathrm{e}^{\mathrm{i}n\theta}. \tag{2.20}$$

This has the well known feature of being independent of N—fluctuations are therefore strongly suppressed. Furthermore, it has been rigorously proved by Johansson [Joh88, Joh98] that the corresponding full distribution of the linear statistic A is given by the central limit-type theorem

$$\Pr(u = A) \underset{N\to\infty}{\sim} \frac{1}{(2\pi\sigma^2)^{1/2}} \, \mathrm{e}^{-(u - \langle A\rangle)^2/2\sigma^2} \tag{2.21}$$

where $\langle A\rangle$ is given by (2.15) with the substitution (2.17), and $\sigma^2 = \mathrm{Var}(A)$ as given by (2.20).

Let us now consider the situation for general μ, $|\mu| < 1$. It is well known [Hua63], and is simple to verify, that under the transformation

$$e^{i\theta_j} = \frac{e^{i\theta_j^{(0)}} - \mu}{1 - \mu^* e^{i\theta_j^{(0)}}} \tag{2.22}$$

(note that the right-hand side has unit modulus and this mapping is one-to-one), the PDF for general μ is transformed into the PDF with $\mu = 0$ according to

$$\prod_{j=1}^{N} \left(\frac{1 - |\mu|^2}{|1 - \mu^* e^{i\theta_j}|^2} \right)^{\beta(N-1)/2+1} \prod_{1 \leqslant j < k \leqslant N} |e^{i\theta_k} - e^{i\theta_j}|^\beta \, d\theta_1 \ldots d\theta_N$$
$$= \prod_{1 \leqslant j < k \leqslant N} |e^{i\theta_k^{(0)}} - e^{i\theta_j^{(0)}}|^\beta \, d\theta_1^{(0)} \ldots d\theta_N^{(0)}. \tag{2.23}$$

This means that the correlation functions for general μ can be obtained from the correlation functions for $\mu = 0$ by applying the inverse of the transformation (2.22)

$$e^{i\theta_j^{(0)}} = \frac{\mu + e^{i\theta_j}}{1 + \mu^* e^{i\theta_j}} \tag{2.24}$$

and noting that

$$d\theta^{(0)} = \frac{(1 - |\mu|^2)}{|1 - \mu^* e^{i\theta}|^2} \, d\theta. \tag{2.25}$$

Hence, from (2.17), we have

$$\rho_{(1)}(\theta) = \frac{N}{2\pi} \frac{(1 - |\mu|^2)}{|1 - \mu^* e^{i\theta}|^2} \tag{2.26}$$

independent of β, and so from (2.15),

$$\langle A \rangle = \frac{N}{2\pi} \int_0^{2\pi} \frac{(1 - |\mu|^2)}{|1 - \mu^* e^{i\theta}|^2} \rho(\theta) \, d\theta. \tag{2.27}$$

For the structure function, substituting (2.24) and (2.25) in (2.18) gives

$$S(\theta_1^{(0)}, \theta_2^{(0)}) \, d\theta_1^{(0)} \, d\theta_2^{(0)} := -\frac{1}{\beta\pi^2} \left(\frac{\partial^2}{\partial\theta_1^{(0)} \partial\theta_2^{(0)}} \log |e^{i\theta_1^{(0)}} - e^{i\theta_2^{(0)}}| \right) d\theta_1^{(0)} \, d\theta_2^{(0)}$$
$$= -\frac{1}{\beta\pi^2} \left(\frac{\partial^2}{\partial\theta_1 \partial\theta_2} \log \left| \frac{e^{i\theta_1} - e^{i\theta_2}}{|1 + \mu^* e^{i\theta_1}||1 + \mu^* e^{i\theta_2}|} \right| \right) d\theta_1 \, d\theta_2$$
$$= -\frac{1}{\beta\pi^2} \left(\frac{\partial^2}{\partial\theta_1 \partial\theta_2} \log |e^{i\theta_1} - e^{i\theta_2}| \right) d\theta_1 \, d\theta_2. \tag{2.28}$$

Thus $S(\theta_1, \theta_2)$ for general μ is identical to $S(\theta_1^{(0)}, \theta_2^{(0)})$ for $\mu = 0$, and consequently

$$\mathrm{Var}(A)^{(0)} = \mathrm{Var}(A). \tag{2.29}$$

This fact illustrates a universality feature of the underlying log-gas: $\mathrm{Var}(A)$ is invariant with respect to the particular one body potential modifying (1.1), provided the corresponding one-body density is a well behaved function.

Finally, we note that (2.29) can be demonstrated via a numerical experiment. The experiment is performed by first generating random unitary matrices with uniform distribution (matrices from the CUE). This can be done by diagonalizing matrices from the GUE (random Hermitian matrices): the matrix of eigenvectors, when multiplied by a diagonal matrix with entries $e^{i\theta_j}$, $(j = 1, \ldots, n)$ where θ_j is a random angle between 0 and

2π with uniform distribution, gives a matrix belonging to the CUE. Next, we calculate the eigenvalues of each matrix (which will have a distribution (1.1) with $\beta = 2$), and transform them according to (2.22) with a specific value of μ. The resulting eigenvalues will have a distribution as on the left-hand side of (2.23). For each set k of eigenvalues $\{e^{i\theta_j}\}_{j=1,\ldots,N}$ we then calculate $A_k := \sum_{j=1}^{N} a(\theta_j)$ for some particular choices of a. From the resulting list of values $\{A_k\}$, the empirical mean and standard deviation are calculated according to the usual formulae. In table 1 we present the result of performing this numerical experiment with N, the dimension of the unitary matrix, equal to 15, and $a(\theta) = \cos j\theta$ $(j = 1, \ldots, 5)$. These empirical values are compared with the theoretical prediction for the variance in the limit $N \to \infty$ as given by (2.20) (note that with $a(\theta) = \cos j\theta$, $a_j = a_{-j} = 1/2$, $a_n = 0$ otherwise; thus (2.20) gives $\mathrm{Var}(A) = j/2$).

Table 1. The second column contains the empirical variance of the quantity $\sum_{j=1}^{N} a(\theta_j)$, with $a(\theta)$ as specified. This was calculated for 500 (15×15) matrices with eigenvalue distribution given by the left-hand side of (2.23) with $\beta = 2$ and $\mu = 0.5$. The final column contains the theoretical variance for the same quantity in the $N \to \infty$ limit.

$a(\theta)$	Empirical $\mathrm{Var}(A)$	Theoretical $\mathrm{Var}(A)$
$\cos\theta$	0.509	0.5
$\cos 2\theta$	0.972	1
$\cos 3\theta$	1.6	1.5
$\cos 4\theta$	1.8	2
$\cos 5\theta$	2.6	2.5

3. Laguerre ensemble with an N-dependent exponent

3.1. Motivation

Recently a number of authors [FS96, GMB96, BFB97] have considered the problem of the distribution of the eigenvalues of the Wigner–Smith matrix $Q = -i\hbar S^{-1}\partial S/\partial E$. Here S refers to the scattering matrix coupled to a perfect lead which supports N channels of the same energy E. For arbitrary N it was found in [BFB97] that for each of the three possible symmetries of S, orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$), the PDF for the reciprocal of the eigenvalues of Q is given by (1.2) with $a = N$. This motivates a study of some of the properties of the distribution functions and fluctuation formulae associated with (1.2) for general $a = YN$, $Y > 0$.

3.2. For $\beta = 1$ and 2

As remarked in the introduction, the PDF (1.2) for $\beta = 1, 2$ and 4 is realized as the eigenvalue PDF of random Wishart matrices $A = X^{\dagger}X$, where X has dimension $(M \times N)$. For $\beta = 1$ and 2, and with a proportional to N, the limiting form of the global density and the statistical properties of the largest and smallest eigenvalues have been extensively studied (see [Ede88] and references therein). In particular, with

$$a = YN \tag{3.1}$$

it is known that the global eigenvalue density is given by

$$\lim_{N\to\infty} \rho(4Nx) = \begin{cases} \dfrac{1}{\pi x}\sqrt{(x - t_1(Y))(t_2(Y) - x)} & t_1(Y) < x < t_2(Y) \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

where

$$t_1(Y) = \tfrac{1}{4}(\sqrt{1+Y} - 1)^2 \qquad t_2(Y) = \tfrac{1}{4}(\sqrt{1+Y} + 1)^2. \tag{3.3}$$

In qualitative terms, the result (3.2) says that the support of the density, which is $(0, 1)$ when $Y = 0$, is repelled from the origin and elongated as Y increases. This is consistent with the log-gas interpretation of (1.2) with a given by (3.1), as then there is an external charge of strength YN placed at the origin. This charge repels the N mobile charges of unit strength away from the origin. The fact that (3.2) is independent of β is also consistent with the log-gas interpretation. In fact, macroscopic electrostatics says that the one-body potential in (1.5) results from a neutralizing background charge density $\rho_b(y)$ according to

$$\frac{x}{2} - \frac{YN}{2}\log x + C = \int_I \rho_b(y)\log|x-y|\,\mathrm{d}y \qquad x \in I \tag{3.4}$$

where I is an interval in $\mathbb{R}^+$. The quantity β does not appear in this equation, so $\rho_b(y)$ is independent of β. However, to leading order the particle density will equal the background density, as in general Coulomb systems to strongly suppress charge fluctuations [Mar88]. The expected independence of (3.2) on β follows.

3.3. *For even* β

For even β, an exact β-dimensional integral representation of the density in the finite system is available [For94, BF97a], which allows the global density limit to be taken explicitly.

Now, in a system of $(N + 1)$ particles, the one-body density $\rho(x)$ in the Laguerre ensemble is given by

$$\rho_{N+1}(x) := \frac{N+1}{Z_{N+1}(a,\beta)}\,\mathrm{e}^{-\beta x/2}x^{\beta a/2}I_N(a,\beta;x) \tag{3.5}$$

where

$$Z_{N+1}(a,\beta) := \prod_{l=1}^{N+1}\int_0^\infty \mathrm{d}x_l\,\mathrm{e}^{-\beta x_l/2}x_l^{\beta a/2}\prod_{1\leqslant j<k\leqslant N+1}|x_k - x_j|^\beta \tag{3.6}$$

$$I_N(a,\beta;x) := \prod_{l=1}^{N}\int_0^\infty \mathrm{d}x_l\,|x-x_l|^\beta\,\mathrm{e}^{-\beta x_l/2}x_l^{\beta a/2}\prod_{1\leqslant j<k\leqslant N}|x_k - x_j|^\beta. \tag{3.7}$$

The normalization (3.6) is a well known limiting case of the Selberg integral, and has the exact evaluation (see e.g. [Ask80])

$$Z_N(a,\beta) = \left(\frac{2}{\beta}\right)^{N(1+\beta a/2)+\beta N(N-1)/2}\prod_{j=0}^{N-1}\frac{\Gamma(1+\beta(j+1)/2)\Gamma(1+\beta(j+a)/2)}{\Gamma(1+\beta/2)}. \tag{3.8}$$

Notice that for even β the integral (3.7) is a polynomial in x. In this case (3.7) has been shown to be expressible in terms of a certain generalized Laguerre polynomial based on Jack polynomials [For94]. Furthermore, this generalized Laguerre polynomial has a different integral representation, which allows the N-dimensional integral (3.7) to be expressed as a β-dimensional integral. This reads [BF97a] as

$$\frac{1}{Z_N(a+2,\beta)}I_N(a,\beta;x) = \frac{1}{Q(a,\beta)}|f(a,\beta;x)| \tag{3.9}$$

where

$$f(a,\beta;x) := \int_{\mathcal{C}_1} \mathrm{d}t_1 \ldots \int_{\mathcal{C}_\beta} \mathrm{d}t_\beta \prod_{j=1}^{\beta} \mathrm{e}^{xt_j} t_j^{-N-3+2/\beta} (1-t_j)^{a+N+2/\beta-1} \prod_{1\leqslant j<k\leqslant\beta} (t_k - t_j)^{4/\beta}. \tag{3.10}$$

The contours of integration in (3.10) must be simple loops which start at $x = 1$ and enclose the origin, and the quantity $Q(a,\beta)$ in (3.9) is chosen so that at $x = 0$, the right-hand side equals 1 (the left-hand side has this property). Also, the modulus sign in (3.9) has been included for convenience to eliminate terms of unit modulus which otherwise occur in $Q(a,\beta)$; this is valid for $x \in \mathbb{R}$ since then the left-hand side is positive. Choosing each $\mathcal{C}_j$ in (3.10) to be the unit circle gives

$$Q(a,\beta) = (2\pi)^\beta M_\beta(a + 2/\beta - 1, N, 2/\beta) \tag{3.11}$$

where

$$\begin{aligned} M_N(a',b',c') &:= \prod_{l=1}^{N} \int_{-1/2}^{1/2} \mathrm{e}^{\pi i\theta_l(a'-b')} |1+\mathrm{e}^{2\pi i\theta_l}|^{a'+b'} \prod_{1\leqslant j<k\leqslant N} |\mathrm{e}^{2\pi i\theta_k} - \mathrm{e}^{2\pi i\theta_j}|^{2c'} \\ &= \prod_{j=1}^{N} \frac{\Gamma(a'+b'+1+(j-1)c')\Gamma(1+c'j)}{\Gamma(a'+1+(j-1)c')\Gamma(b'+1+(j-1)c')\Gamma(1+c')} \end{aligned} \tag{3.12}$$

(the integral M_N is due to Morris [Mor82]).

Our interest is in the large-N asymptotic form of $\rho_{N+1}(4Nx)|_{a=YN}$, which from (3.5) and (3.9) requires the large-N form of $f(YN,\beta;x)$. The necessary technique is a generalized saddle point analysis as introduced in [For94] and presented in detail in [BF97a]. The saddle points occur at the stationary points of

$$N(4xt_j - \log t_j + (Y+1)\log(1-t_j)) \tag{3.13}$$

which is the N-dependent term of the integrand of $f(YN,\beta;x)$ when expressed as an exponential. A simple calculation finds that there are two stationary points, t_+ and t_- say, given by

$$t_\pm = \frac{x - Y/4 \pm ((x-Y/4)^2 - x)^{1/2}}{2x}. \tag{3.14}$$

Note that $t_+ = t_-^*$ for

$$(x - Y/4)^2 - x < 0. \tag{3.15}$$

Assuming (3.15), following the strategy of [BF97a], the leading large-N asymptotic behaviour is obtained by deforming $\beta/2$ of the contours through t_+, and the remaining $\beta/2$ of the contours through t_- (this introduces a factor of $\binom{\beta}{\beta/2}$ to account for the number of ways of dividing the β contours into these two classes). The calculation now proceeds in a conventional way, with the exponent (3.13) being expanded about $t_\pm$ to second order, and the N-independent terms in the integrand replaced by their value at t_+ or t_- as appropriate. After this step we have

$$\begin{aligned} |f(YN,\beta;x)| &\sim \binom{\beta}{\beta/2} |g_2(t_+,Y)|^\beta \left| \int_{-\infty}^{\infty} \mathrm{d}t_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}t_{\beta/2}\, \mathrm{e}^{-(N/2)g_1(t_+,Y)(t_1^2+\cdots+t_{\beta/2}^2)} \right. \\ &\quad \left. \times \prod_{1\leqslant j<k\leqslant\beta/2} |t_k - t_j|^{4/\beta} \right|^2 = \binom{\beta}{\beta/2} |g_2(t_+,Y)|^\beta |Ng_1(t_+,Y)|^{-\beta+1} (V_{\beta/2}(2/\beta))^2 \end{aligned} \tag{3.16}$$

where

$$g_1(t_+, Y) := \frac{1}{t_+^2} - \frac{1+Y}{(1-t_+)^2} \tag{3.17}$$

$$g_2(t_+, Y) := \mathrm{e}^{N(4xt_+ - \log t_+ + (Y+1)\log(1-t_+))} t_+^{-3+2/\beta} (1-t_+)^{2/\beta - 1}(t_+ - t_-) \tag{3.18}$$

$$\begin{aligned} V_N(c) &:= \int_{-\infty}^{\infty} \mathrm{d}t_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}t_N \, \mathrm{e}^{-(1/2)(t_1^2 + \cdots + t_N^2)} \prod_{1\leqslant j<k\leqslant N} |t_k - t_j|^{2c} \\ &= (2\pi)^{N/2} \prod_{j=0}^{N-1} \frac{\Gamma(1+c(j+1))}{\Gamma(1+c)}. \end{aligned} \tag{3.19}$$

The equality in (3.16) follows from a simple change of variables, while (3.19) is known as Mehta's integral, and can be evaluated as a limiting case of the Selberg integral [Ask80] (for a direct evaluation see [Eva94]).

From definition (3.14) we have that

$$|x_+|^2 = \frac{1}{4x} \qquad |1-x_+|^2 = \frac{1+Y}{4x} \qquad |t_+ - t_-|^2 = \frac{1}{x^2}(x - (x - Y/4)^2)$$

and use of these results in formulae (3.17) and (3.18) defining g_1 and g_2 shows that

$$\frac{|g_2(t_+, Y)|^\beta}{|g_1(t_+, Y)|^{\beta-1}} = \mathrm{e}^{2\beta N(x-Y/4)}(1+Y)^{(1+Y)N\beta/2}(4x)^{-YN\beta/2}(1+Y)^{1/2}\frac{1}{x}(x-(x-Y/4)^2)^{1/2}. \tag{3.20}$$

Furthermore, from (3.8)

$$\begin{aligned} \frac{Z_N(a+2,\beta)}{Z_{N+1}(a,\beta)} &= \left(\frac{2}{\beta}\right)^{N\beta-(\beta/2)} \frac{\Gamma(1+(\beta/2))}{\Gamma(1+(\beta/2)(N+1))} \\ &\quad \times \frac{\Gamma((\beta/2)(a+N+1)+1)}{\Gamma((\beta/2)a+1)\Gamma((\beta/2)(a+1)+1)} \end{aligned} \tag{3.21}$$

while using (3.12) in (3.11) and by comparison with (3.19) shows that

$$\frac{1}{Q(a,\beta)} = \frac{1}{(2\pi)^{\beta/2} V_\beta(2/\beta)} \prod_{j=1}^{\beta} \frac{\Gamma(a+(2/\beta)j)\Gamma(N+1+(2/\beta)(j-1))}{\Gamma(a+N+(2/\beta)j)} \tag{3.22}$$

and using (3.19) gives

$$\frac{(V_{\beta/2}(2/\beta))^2}{V_\beta(2/\beta)} = (\beta/2)^{\beta/2}\frac{\Gamma(1+(\beta/2))}{\Gamma(1+\beta)}. \tag{3.23}$$

With (3.9) substituted in (2.23), the remaining task is to use Stirling's formula to compute the leading large-N asymptotic behaviour of (3.21) and (3.22) with $a = YN$. Performing this task, and substituting the resulting expression together with (3.20) and (3.23) in (3.9), shows that for x such that (3.15) is true

$$\lim_{N\to\infty} \rho_{N+1}(4Nx)|_{a=YN} = \frac{1}{2\pi x}(x-(x-Y/4)^2)^{1/2}. \tag{3.24}$$

Outside the interval (3.15), i.e. outside $x \in [(1/2)(1+(Y/2)-\sqrt{1+Y}), (1/2)(1+(Y/2)+\sqrt{1+Y})]$, this limit must vanish. This is seen from the fact that the density is positive, and must satisfy the normalization

$$\int_0^\infty \rho_{N+1}(4Nx)\,\mathrm{d}x \sim \frac{1}{4}$$

which is satisfied by the right-hand side of (3.24). The result (3.24) for the scaled density, established for even β, is identical to the result in (3.2) known for $\beta = 1$ and 2, as expected.

4. The distribution functions in the neighbourhood of the smallest eigenvalue

4.1. The n-point distributions

For fixed N and $\beta = 1, 2$ or 4, the exact expressions for the general n-point distribution function in the Laguerre ensemble are known [Bro65, NW91, NF95]. With $a = YN$ and N large, it is natural to move the origin to the (mean) location of the smallest eigenvalue, and to scale the eigenvalues so that the mean spacing near the spectrum edge is $O(1)$. One anticipates that the limiting n-point distribution function will correspond to the n-point distribution function for the so-called soft edge, which is the edge of the spectrum for the Gaussian random matrix ensemble, with the eigenvalues appropriately scaled. For the one-body densities and $\beta = 1$ and 2, this has been established by Feinberg and Zee [FZ97].

In quantitative terms, we expect that for appropriate $\nu(N)$ independent of β

$$\lim_{N\to\infty} (\nu(N))^n \rho_{(n)}(N(1-\sqrt{1+Y})^2 - \nu(N)x_1, \ldots, N(1-\sqrt{1+Y})^2 - \nu(N)x_n) \\ = \rho^{\rm soft}_{(n)}(x_1, \ldots, x_n). \tag{4.1}$$

On the left-hand side, $\rho_{(n)}$ refers to the n-point distribution function for the Laguerre ensemble with $a = YN$. Note from (3.2) and (3.24) that $N(1-\sqrt{1+Y})^2$ is the location of the smallest eigenvalue (to leading order in N). On the right-hand side

$$\rho^{\rm soft}_{(n)}(x_1, \ldots, x_n) := \lim_{N\to\infty} \left(\frac{1}{2^{1/2}N^{1/6}}\right)^n \\ \times \rho_{(n)}\left((2N)^{1/2} + \frac{x_1}{2^{1/2}N^{1/6}}, \ldots, (2N)^{1/2} + \frac{x_n}{2^{1/2}N^{1/6}}\right) \tag{4.2}$$

where here $\rho_{(n)}$ refers to the n-point distribution function for the Gaussian ensemble defined by the eigenvalue p.d.f

$$\frac{1}{C}\prod_{l=1}^N {\rm e}^{-\beta x_l^2/2} \prod_{1\leqslant j<k\leqslant N} |x_k - x_j|^\beta \tag{4.3}$$

and $(2N)^{1/2}$ is the leading-order location of the largest eigenvalue. In this section (4.1) will be explicitly verified for $\beta = 2$, with the quantity $\nu(N)$ shown to be given by

$$\nu(N) = N^{1/3} \frac{2^{1/3}(\sqrt{1+Y}-1)^2}{(Y^2 - (2+Y)(\sqrt{1+Y}-1)^2)^{1/3}}. \tag{4.4}$$

Once $\nu(N)$ has been determined, the validity of (4.1) for $n = 1$ can be established by matching the one-body density in the neighbourhood of the smallest eigenvalue implied by (3.24) with the asymptotic behaviour [For93]

$$\rho^{\rm soft}_{(1)}(x) \underset{x\to-\infty}{\sim} \frac{\sqrt{|x|}}{\pi} \tag{4.5}$$

(this idea is motivated by a similar procedure used in [Nis96, KF97a]). Now the result in (3.24) implies that for all β

$$\rho_{(1)}(N(1-\sqrt{1+Y})^2 - \nu(N)x) \underset{\substack{N\to\infty \\ x\to-\infty}}{\sim} \frac{\sqrt{|x|}}{\pi} \frac{(1+Y)^{1/4}}{(1-\sqrt{1+Y})^2}\sqrt{\frac{\nu(N)}{N}} \tag{4.6}$$

valid for $0 \ll x \ll N/\nu(N)$. Substituting (4.5) and (4.6) in (4.1) with $n = 1$, and using (4.4), shows that (4.1) is satisfied provided

$$(1+Y)^{1/4} \frac{2^{1/2}(\sqrt{1+Y}-1)}{((2+Y)(Y^2 - \sqrt{1+Y}-1)^2)^{1/2}} = 1 \tag{4.7}$$

which is readily verified.

In preparation for verifying (4.1) for $\beta = 2$ and general n, we first recall some formulae particular to that coupling. For the Laguerre ensemble (1.2) we have [Bro65]

$$\rho_{(n)}^{\rm soft}(x_1, \ldots, x_n) = \det[P_N(x_j, x_k)]_{j,k=1,\ldots,n} \tag{4.8}$$

where with $L_n^a(x)$ denoting the Laguerre polynomial of degree n

$$\begin{aligned} P_N(x, y) &:= (xy)^{a/2}\, e^{-(x+y)/2} c_N \frac{L_N^a(x)L_{N-1}^a(y) - L_N^a(y)L_{N-1}^a(x)}{x-y} \\ &:= (xy)^{a/2}\, e^{-(x+y)/2} \frac{c_N}{N+a} \frac{L_N^a(x) y L_N^{a\prime}(y) - L_N^a(y) x L_N^{a\prime}(x)}{x-y} \\ c_N &= \frac{\Gamma(1+N)}{\Gamma(a+N)}. \end{aligned} \tag{4.9}$$

Furthermore, we know that [For93]

$$\rho_{(n)}^{\rm soft}(x_1, \ldots, x_n) = \det[K^{\rm soft}(x_j, x_k)]_{j,k=1,\ldots,n} \tag{4.10}$$

where, with $\mathrm{Ai}(x)$ denoting the Airy function,

$$K^{\rm soft}(x, y) := \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y}. \tag{4.11}$$

Comparison of (4.10) and (4.8), and the fact that (4.1) is valid for $n = 1$ once (4.4) is established shows that to verify (4.1) it suffices to establish the asymptotic formula

$$x^{NY/2}\, e^{-x/2} L_N^{YN}(x)\big|_{x=N(1-\sqrt{1+Y})^2-\nu(N)X} \underset{N\to\infty}{\sim} k_N(Y)\mathrm{Ai}(X) \tag{4.12}$$

with $\nu(N)$ given by (4.4) and μ fixed. There is no need to specify $k_N(Y)$, as its value is uniquely determined by (4.5) and (4.6). Indeed, substituting (4.12) in (4.9), then substituting the resulting expression in (4.8) with $n = 1$, setting $x = y$ and comparing with (4.1) for $x \to -\infty$ shows that

$$(k_N(Y))^2 = \frac{1}{(1-\sqrt{1+Y})^2} \frac{\nu(N)}{c_N}. \tag{4.13}$$

In table 2 we give the numerical value of the ratio of the left-hand side to right-hand side of (4.12) with $k_N(Y)$ given by (4.13) for various values of X, Y, N.

Table 2. Numerical values of the ratio of the left-hand side to right-hand side of (4.12) for various values of N and (X, Y).

Value of N	Value of (X, Y): $(0, 1)$	$(1, 2)$	$(-1, 2)$
50	1.0426	1.0567	0.9929
60	1.0402	1.0556	0.9938
70	1.0383	1.0544	0.9944
80	1.0367	1.0533	0.9949
90	1.0353	1.0523	0.9953

The asymptotic formula (4.12) can be derived by utilizing the fact that $y = e^{-x/2}x^{(\alpha+1)/2}L_N^\alpha(x)$ satisfies the second-order differential equation

$$y'' + \left(\frac{2N+\alpha+1}{2x} + \frac{1-\alpha^2}{4x^2} - \frac{1}{4}\right) y = 0. \tag{4.14}$$

Substituting $\alpha = YN$, $x = N(1-\sqrt{1+Y})^2 - \nu(N)X$, shows that for large N (4.14) reduces to

$$y'' - \frac{1}{2(1-\sqrt{1+Y})^6}(Y^2 - (2+Y)(1-\sqrt{1+Y})^2)\frac{(\nu(N))^3}{N}xy = 0. \quad (4.15)$$

With $\nu(N)$ given by (4.4), this equation reads $y'' - xy = 0$ and its unique solution which decays as $X \to -\infty$ is $y = k_N(Y)\mathrm{Ai}(X)$, thus establishing (4.12).

4.2. Distribution of the smallest eigenvalue

In principle, knowledge of the n-point distributions allows the calculation of other statistical quantities such as the distribution of the smallest eigenvalue. This together with the above results implies that, after appropriate change of origin and scale, the PDF of the smallest eigenvalue in the Laguerre ensemble with $a = YN$ equals, in the $N \to \infty$ limit, the PDF for the largest (or equivalently smallest) eigenvalue in the Gaussian ensemble (4.3). For $\beta = 2$, this can be explicitly verified from the nonlinear equations characterizing the respective PDFs due to Tracy and Widom [TW94a, TW94b].

Let $E(x)$ denote the probability that the interval $(0, x)$ in the Laguerre ensemble (1.2) with $\beta = 2$ contains no eigenvalues, and let

$$\sigma(x) = x\frac{\mathrm{d}}{\mathrm{d}x}\log E(x). \quad (4.16)$$

Then it has been derived in [TW94b] that $\sigma(x)$ satisfies the nonlinear equation

$$(x\sigma'')^2 = 4x(\sigma')^3 + \sigma^2 + (2a + 4N - 2x)\sigma\sigma' + (a^2 - 2ax - 4Nx + x^2)(\sigma')^2 - 4\sigma(\sigma')^2. \quad (4.17)$$

Also, let $\tilde{E}(x)$ denote the probability that there are no eigenvalues between (x, ∞) in the (infinite-dimensional) Gaussian ensemble (4.3) with coordinates as in (4.2). Then it was derived in [TW94a] that the quantity

$$R(x) := \frac{\mathrm{d}}{\mathrm{d}x}\log\tilde{E}(x) \quad (4.18)$$

satisfies the nonlinear equation

$$(R'')^2 + 4R'((R')^2 - xR' + R) = 0. \quad (4.19)$$

As the PDF for the smallest (largest) eigenvalue is simply related to $E(x)$ ($\tilde{E}(x)$) by differentiation, we see that the differential equation (4.17) characterizes the PDF for the smallest eigenvalue in the Laguerre ensemble with $\beta = 2$, while (4.19) characterizes the limiting form of the PDF of the eigenvalue at the spectrum edge of the Gaussian ensemble. (Of course boundary conditions must be specified; these follow from the small (large) x behaviour of the density.) The results of the previous subsection imply that

$$\lim_{N\to\infty} E(N(\sqrt{1+Y} - 1)^2 - \nu(N)x)\big|_{a=YN} = \tilde{E}(x). \quad (4.20)$$

Indeed, a straightforward calculation using the explicit form (4.4) of $\nu(N)$ shows that after changing variables $x \mapsto N(\sqrt{1+Y} - 1)^2 - \nu(N)x$ in (4.17) and introducing the function

$$\tilde{\sigma}(x) := \frac{1}{\nu(N)}\sigma(N(\sqrt{1+Y} - 1)^2 - \nu(N)x) \quad (4.21)$$

the differential equation (4.17) reduces down to (4.19) with $R = \tilde{\sigma}$. This is precisely what is required by (4.20).

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